Hermann Weyl on Minkowskian space-time and Riemannian geometry

1. Minkowski's space-time

I want to emphasize in the following that Weyl has taken for granted the Minkowskian world picture as far as Special Relativity is concerned, that is the four-dimensional representation of the space-time continuum. Weyl uses freely the Minkowskian vocabulary of world, world-lines and world-points – or even the Minkowskian "*Substanz*" to designate matter in his philosophical works, for example, *Philosophy of Mathematics and Natural Science* [6]. But, of course, he resorts to Riemannian geometry when it comes to the geometrical structure of the universe as a whole in his unified field theory.

Weyl credits Minkowski for having recognized that:

> The fundamental equations for moving bodies are determined by the principle of relativity if Maxwell's theory for matter at rest is taken for granted ([6], 96)

He says also referring to Minkowski's 1907 (see [2] paper "Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körper" or "The fundamental equations for electromagnetic processes in moving bodies", that :

> The adequate mathematical formulation of Einstein's discovery was first given by Minkowski: to him we are indebted for the idea of four-dimensional world-geometry…([6], 173)

Minkowski distinguishes in the aforementioned paper the theorem of relativity from the principle of relativity ; the first is purely mathematical in terms of the covariance of Lorentz transformations and the second, the principle of relativity allows, in Minkowski's words, for the derivation of the laws of mechanics solely from the principle of the conservation of energy. The language here is still of space-time vectors and does not anticipate on the 1908 paper on "*Raum und Zeit*" (in [2]) where the vocabulary of world-points and world-lines is canonized and serves as the basic ingredient for a graphic representation, as Minkowski says, of the group of spatio-temporal transformations. Minkowski then suggests that the terminology for the postulate of relativity is rather dull – "matt" in German – when one wants to stress the invariance properties of the group of the transformations and he comes up with his "postulate of the absolute world" [*Postulat der absoluten Welt*]. Let me point out that at one time Einstein himself wanted to rename relativity theory as invariance theory.

My contention is that Minkowski's idiom has essentially a mathematical meaning as a representational means for the spatio-temporal structure of the physical world. And I assume that Weyl has not interpreted the Minkowskian mathematical picture otherwise. I shall give some reasons to substantiate that claim in the following.

Minkowski's work on physical problems (hydrodynamics, capillarity, etc. and on Relativity Theory) is marginal compared to his endeavour in number theory, geometry and especially what Minkowski called the geometry of numbers. It is in the geometrical representation of number-theoretic relations that Minkowski introduces the notion of "*Zahlengitter*" or number grids. Minkowski defines a three-dimensional number grid as a geometrical representation of three integers in rectangular coordinates (cf. "Über Geometrie den Zahlen" in *Gesammelte Abhandlungen* or *Collected Papers*, [2], 264-265).

This geometry of numbers has an arithmetical core, while geometry has an intuitive appeal. A number grid is most important for the representation of the volume of a body and its fundamental arithmetical property is the generalization of the length of a straight line into the principle that in a triangle the sum of the lengths of two sides is never smaller than the length of the third side. As a special case, one easily points to the Pythagorean theorem for right triangles:

$$c^2 = a^2 + b^2$$

which is at the foundation of the differential form

$$ds^2 = dx_1^2 + dx_2^2$$

in Euclidean coordinates. Weyl will point out that the differential form ds^2 is not only the simplest, but also the most appropriate one for the classification of possible geometries since the positive quadratic form generates all linear transformations of the variables involved. Of course, the ds^2 is the fundamental (quadratic) form

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

for the invariant metric element in special Relativity. Let me remark that if one lifts the quadratic restriction, one gets Finsler spaces which are akin to Riemannian spaces. I do not need to mention that Minkowski has devoted much of his work to the theory of quadratic forms, *i.e*. homogeneous polynomials of the second degree which where the main object of study in number theory from Gauss to Kronecker. Rather than elaborate on this, I can only mention that number grids are clearly connected to what we now call Minkowski diagrams.

In his diagrams, Minkowski depicts grid points and intersection points for the approximation of a real number (or quantity) by rational numbers or diagonal chains that illustrate the fact that in a triangle the sum of the lengths of two sides cannot be smaller than the length of the third one. Here entire rational functions are used as denominators of continued fractions. What we have is an intuitive representation of solutions of a real inequality in terms of integers without common divisors.

Those diagrams do not differ essentially from the ones which are to be found in the paper "*Raum und Zeit*". What is represented here as a grid point is the notion of an arbitrary point like charge or electron or scalar potential in a light cone. The world postulate or the postulate of the absolute world is nothing more than the totality of those grid points along grid lines in a gridded universe without an ontological import.

What philosophical conclusions can be drawn from my analysis? For the ontology of space-time, a gridded universe could be empty or devoid of any substance or *Substanz* if we want to use Minkowski's word for matter. Since the postulate of the absolute world can be given a purely mathematical signification in virtue of its arithmetico-geometrical foundations, I would grant it, following Weyl, only a transcendantal status, that is the status of an a priori structure in the theoretical construction of the world, as Weyl puts it in his *Philosophy of Mathematics and Natural Science*. Minkowski diagrams belong to the analytical apparatus, "*der analytische Apparat*" as Hilbert termed it in his work on the foundations of Quantum Mechanics (see [1])[1]; the diagrams do not belong to a model of the physical world. From that standpoint, a realist interpretation of the Minkowskian world view is thereby excluded on Minkowski's own terms. Although Minkowski states that his intuitions of space and time rest on the solid ground of experimental physics, their validity must be sought in the mathematical justification of the intuitive content. Behind or below the physical geometry, "*physikalische Geometrie*" as Helmholtz called it, lies a geometry of numbers, the heart of which is arithmetic or number theory or simply number, thus coming back to Pythagoras who is the originator of the whole story and his (or their) theorem the cornerstone of the entire edifice.

2. Riemann's geometry

Hermann Weyl's foundational stance in mathematics is many-faceted. From the predicative mathematics of *Das Kontinuum* (1917) to the constructivist Kroneckerian point of view of *Algebraic theory of numbers* (1939), Weyl has sustained a continuous effort to defend a constructivist philosophy in mathematics, physics and science in general. I do not want to put any emphasis on his philosophical inspirations from Kant and Fichte to Husserl and Brouwer. I shall rather insist on the motivations internal to his

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[1] I refer here to the paper by Hilbert, von Neumann and Nordheim, "Ueber die Grundlagen der Quantenmechanik", *Mathematische Annalen*, 98, 1-30.

mathematical work and particularly on his appraisal of Riemannian geometry, which has played a major role in Weyl's theoretical endeavours.

I recall the main ideas of Riemann's "Ueber die Hypothesen welche der Geometrie zu grunde liegen", "On the hypotheses of the foundations of geometry" (see [3]). The concept of an n-dimensional manifold or space (*Mannigfaltigkeit*) is a topological one, that is, it is to be treated with continuous (real and complex) functions. As a metric space, it is endowed with the metric invariant ds^2 and here Riemann refers to Gauss' theory of curved surfaces.

The fundamental quantity for any surface is the element of arc length

$$ds^2 = dx^2 + dy^2 + dz^2.$$

Gauss gives a general form to this notion of distance by introducing parameters u and v to represent the coordinates (x, y, z) of any point on the surface and obtains

$$ds^2 = e(u,v)\, dv^2 + 2f(u,v)\, du\, dv + g(u,v)\, dv^2$$

(where e, f, g are functions of the parameters u and v). The general formula for the ds^2 is

$$ds^2 = g_{mn}\, dx_m\, dx_n$$

where dx_m and dx_n are the infinitesimal transformations of the parameterized curves on the surface; and the g_{mn} are the quantities that depend on those curves. The curvature of a surface is then determined from this fundamental quadratic form; this is the point of departure of Riemann's work. The local or infinitesimal approach favoured by Riemann allows for a generalization of Gauss' surfaces into the n-dimensional manifold with Weyl's fundamental tensor

$$ds^2 = g_{uv}\, dx^u\, dx^v \qquad (\text{for } u, v = 1, \ldots, n)$$

which controls the behaviour of the geodesics for a point and a direction in an n-dimensional manifold.

The local infinitesimal point of view insures that the Pythagorean theorem is valid in the infinitely small and in his remarks on Riemann's lecture, Weyl insists again on the fact that the differential form ds^2 is not only the simplest, but also the most appropriate one for the classification of possible geometries, since the positive definite quadratic form generates all linear transformations of the variables involved as I have said above. Group theory enters the picture as, for example, in the case of the alternate subgroup of even permutations of a given group for the distinction between geometries of various dimensions. But Pythagorean-Riemannian geometry is a special case and Weyl adds that one has to supplement the names of Lie and Helmholtz in order to have infinitesimal mobility (*Beweglichkeit*) in an n-dimensional manifold.

The infinitesimal Lie group of rotations obeys the same positive definite quadratic form and also validates the Pythagorean theorem in the infinitely small. Weyl points out that his solution to the new problem of space implied in Relativity Theory, as he says, has urged him to construct a different solution, that is the affine geometry of parallel displacement which Weyl will put to use in his *Raum, Zeit, Materie*, “Space, Time, Matter” of 1918 (see [5]).

3. Riemann’s “ hypotheses”

Weyl puts the emphasis on Riemann’s achievement for being the first to define the idea of a physical geometry. Against all former mathematicians and philosophers, Weyl says it is Riemann who maintains that the metric field is determined by the material content of the world and has the same status as the electromagnetic field. A Riemmannian manifold has constant curvature and in such a manifold free mobility is guaranted by the Lie group of rotations. But this again implies that the physical content of the space participates in the metric in such a way that it is the distribution of matter which defines ultimately the metric field, an idea rediscovered by Einstein, Weyl tells us. It is not doubtful that Riemann had such a scientific objective. In his *Pariser Arbeit*, for example, he deals with the linear transformations involved in second-order differential equations to tackle the problem of an isothermic system again in terms of a fundamental quadratic form. In many of his works, Riemann could be considered as a mathematical physicist, as Weyl suggests, and I would suggest that he could be considered also as a philosopher of science, at least to the extent that he has expressed himself even so briefly on matters philosophical. Take, for instance, Riemann’s notion of hypothesis.

As a student of the Kantian Herbart, who defended a kind of physical *a priori*, Riemann distinguishes his sense of hypothesis from the Newtonian notion found in *Philosophiae naturalis principia mathematica* :

> « Quidquid enim ex Phaenomenis non deducitur,
> Hypothesis vocanda est ».

Riemann prefers to define hypothesis as anything (concepts and principles) that goes beyond phenomena, and not as figments of one's imagination:

> " Man pflegt jetzt unter Hypothese Alles zu der Erscheinungen
> Hinzugedachte zu verstehen" (Riemann, 525).

For Riemann, hypotheses and facts <*Thatsachen*> share a common status as conceptual structures that supervene on phenomena; the law of inertia, for example, as a law of motion is a hypothesis, not different from an axiom, since it belongs to the transempirical or empirical *a priori* realm, while an axiom in the traditional sense should be analytical. By the same token, one could say that facts are the same as hypotheses and Helmholtz' work «*Ueber die Thatsachen die der Geometrie zugrunde liegen*» stands in perfect continuity with Riemann's notion of hypothesis. As a matter of fact, Riemann states the law of inertia as a counterfactual conditional, much in the manner of later logical positivism:

> "If there were only a single material point in the world
> moving in space at a definite velocity, it would always
> move with the same velocity"(Riemann, 525).

Riemann's notion of hypothesis reaches beyond Kant to Descartes and Aristotle. One could justify the hypothetico-deductive method by quoting Descartes who says in his *Principles of Philosophy* (IV, 204):

> "For the things that our senses cannot reach, it suffices
> to explain how they could be".

Descartes adds : it is exactly what Aristotle has done and he refers here to Aristotle's *Meteors* (I,7):

> " If it is true that for phenomena which escape our senses

> we deem to have given a rational explanation by ascending to their possible causes (το δυνατον αυαγαγωμεν), it is certainly true for those phenomena which we now study (the meteors)”.

This anagogical way has been called abduction by Peirce and it is certainly part of what has been called « inference to the best explanation ». In any case and without pouring too much philosophy into Riemann’s scarse remarks, one should not neglect the foundational motivations in Riemann’s mathematical work.

4. Physics

In the Special Theory of Relativity, the Mindowskian four-dimensional manifold is Euclidean, but the quadratic form here is not positive definite, it has an index of inertia of 1; it is then called semi-Riemannian as in the case of Special Relativity

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2\, dt^2$$

while in the General Theory, the manifold is Riemannian, since the gravitational material content of the world forces a curvature onto the structure of the metric field.

Incidentally, Riemann points out in his lecture that such a curved space or spherical universe could be unlimited in extension without being infinite from the point of view of measurement. In the case of a space with positive curvature, however small it is, an unlimited spherical surface is necessarily finite ([3], 284).

In his paper “*Reine Infinitesimalgeometrie*”, Pure infinitesimal Geometry (1918), Weyl declares that Riemann is not always consequent, for infinitesimal geometry forbids to relate two finite elements when they are at an arbitrary finite distance from each other, as Riemann supposes. The relation is possible only in infinitesimal affine geometry for infinitely near points. As Weyl puts it in *Space, Time, Matter*, it is :

> "The principle of gaining knowledge of the external world from the behaviour of its infinitesimal parts. In a Riemannian geometry, geodesics have a stationary length, which is not the case in affine geometry" ([5], 193)

I want to comment briefly on the fate of Weyl's own geometrical idea of affine geometry, an idea which has been instrumental in his unified field theory of electromagnetism and gravitation in *Raum, Zeit, Materie*.

Weyl indicates that an affine relationship or transfer of direction and not only of length in infinitesimal parallel displacements is inherent in metric space. The metric field is still dominated by the quadratic form ds^2 while the electromagnetic field is endowed with a linear fundamental form $\varphi_i \, dx_i$ which has its origin in the metric field. Weyl will admit later that the gauge invariance between the electromagnetic and the gravitational fields was to be grounded on a more fundamental quantum theory. He had also introduced a cosmological constant as Einstein did, but Weyl thought that Einstein's lambda (λ) was arbitrary, as Einstein later admitted, for it was meant only to insure spatial closure of a stationary universe, while Weyl claims an inner necessity for his own λ, because his affine geometry accounts for the electromagnetic potentials and fully obey Maxwell's equations which demand the equilibrium state of the total mass of a closed universe. An inner necessity however which finally vanished like Einstein's blunder. The fact that it is revived today in supersymmetry theories is evidence that blunders have their own constancy if not their own cosmology. There remains though Weyl's fundamental idea of a unified theory based on gauge invariance which is still a major building block in contemporary physics.

5. Conclusion

I want to finish with some remarks on Weyl's foundational stance. From his early mathematical work *Das Kontinuum* and *Raum, Zeit, Materie*, up to his last works including his philosophical papers, Weyl has constantly held a philosophical viewpoint that one can certainly term as constructivist. Although, he has at times admitted that he has used freely transcendantal methods in his representation theory of continuous semi-simple groups of linear transformations (cf. *Theorie der Darstellung kontinuierlicher halbeinfacher Gruppen durch lineare Transformatronen*) and although he has also said that he has rejected all Puritan doctrines – understanding here constructive or elementary methods – to the profit of the analytic method (cf. *The Classical Groups. Their Invariants and their Representations*, 1939), he has constantly put the emphasis on the algebraic constructions, which are direct methods, as he says. If the analytical means are more elegant, they must give rise to results that can be algebraically reconstructed. In the same way, the use of analogies between the finite case and the infinite one is only of limited interest (for the sake of harmony) in algebraic number theory – the same is true in p-adic analysis where one will want to reduce the infinite prime spots as one does for the points at infinity in projective geometry. And of course, in pure number theory, elementary or constructive methods must be privileged simply because arithmetic is discrete in its essence. Decision procedures must terminate in dealing with finite sets of integers, that is finite fields. In the end, we have the idea of an internal logic in Weyl's life-long endeavour in mathematics, physics and philosophy when he says:

> Each field of knowledge, when it crystallizes into a formal theory, seems to carry with it its intrinsic logic which is part of the formalized symbolic system and this logic will, generally speaking, differ in different fields ([4], 705)

The context is here quantum logic, that is the internal logic of quantum mechanics, but it is, I think, relevant for Weyl's thought *in toto*, for he has always stressed that science and philosophy are guided by the theoretical construction of the world (cf. *Philosophy of Mathematics and Natural Science*).

The unified world picture is but the end result of a diversified approach. The various avenues that lead to the ultimate goal of theoretical construction may take different forms with their own internal logic. Such a logic is not a universal grammar, that is, a central facility serving all comers, as Quine's saying goes. The internal logic of theoretical construction requires a selection of goods according to the specific needs of particular consumers.

It cannot be denied that Weyl has stressed the symbolic construction of the world, as for example in his *Philosophy of Mathematics and Natural Science*. In such a task, mathematics and physics are intimately related :

> A truly realistic mathematics should be conceived in line with physics, as a branch of the theoretical construction of the one real world, and should adopt the same sober and cautioned attitude toward hypothetic extensions of its foundations as is exhibited by physics.

One also recalls the <*gegenseitige Durchdringung*> or “multiple connectedness” of his philosophical, mathematical and physical interests, as he puts it in the preface of the first edition of *Raum, Zeit, Materie* (1918). But beyond the general philosophical attitude which one could readily characterize as Kantian (or neo-Kantian or Husserlian), there remains the constant concern for the constructivist foundations of science.

As a final remark, I would like to point out that most philosophers and historians of science are not sufficiently aware of the constructive dimension in the theoretical work of the mathematician or the physicist. Take Riemannian geometry as an example. I have insisted on the centrality of quadratic forms in the construction of the concept of space, starting from the Pythagorean theorem to Gauss’s curved surfaces, Riemann’s n-dimensional manifolds and Minkowski’s number grids and diagrams. Following Weyl, we view infinitesimal differential geometry as applying the Pythagorean theorem in an infinitesimal setting where the arithmetic of quadratic forms or homogeneous polynomials of the second degree still hold. The invariant quadratic form ds^2 is central, simply because it is the fundamental measure of areas of surfaces as in the Pythagorean theorem. Compare this to the central role played by Pythagorean triplets (again from the Pythagorean theorem) in number theory from Fermat to contemporary arithmetic-algebraic geometry (for this, see [1]).

The geometrical model of the arithmetic form is the metric invariant ds^2 of Special Relativity which finds its way as the metric tensor in General Relativity. But differential forms (linear, quadratic and of higher degree) as homogeneous polynomials pervade also Quantum Mechanics from Schrödinger's equation to Quantum Field Theories. Those constructions emanate from what Kronecker called "general arithmetic" <*allgemeine Arithmetik* > (covering most of algebra) together with the corresponding topological invariants. Those forms are the true a priori constituents or conditions of reality – "*Realitätsbedingungen*" as Hilbert called them – of a physical theory.

To take one further example, although I agree in general with Michael Friedman's analysis of the philosophical foundations of Einstein's General Relativity (as formulated in his recent *Dynamics of Reason*, CSLI Publications, Stanford, 2001), I cannot but find intriguing the idea that Einstein's principle of equivalence was the only means of realizing a physical model for Riemannian geometry through the Helmholtz-Lie theorem on free mobility. Affine geometry and Weyl's field theory which incorporate of course the equivalence principle (and even Mach's principle) are not discussed by Friedman, but from Weyl's point of view his solution to the space problem is better than Einstein's. As a matter of fact, Einstein's solution is but a particular model of the analytical apparatus that encompasses the topological, differential and metrical structure of the model. Since the analytical apparatus is not canonical or categorical, there is no principal model, only a standard model alongside a multiplicity of non-standard ones. To pick the one Einsteinian model as the standard model might be a matter of convention as Poincaré proposed, a proposal which is not rejected by Friedman.

But, one must remember though that conventions are also constructions for Poincaré and among those, there are good ones and bad ones. And of course, the success of a physical theory depends on the choice of a good convention or a better construction.

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