Constructive truth and certainty in logic and mathematics

1. INTRODUCTION

The theme « Truth and Certainty » is reminiscent of Hegel's dialectic of <*Wahrheit und Gewissheit* > prominent in the *Phänomenologie des Geistes*, but I want to treat it from a different angle in the perspective of the constructivist stance in the foundations of logic and mathematics. Although constructivism stands in opposition to mathematical realism, it is not to be considered as an idealist alternative in the philosophy of mathematics. It is true that Brouwer's intuitionism, as a variety of constructivism, has idealistic overtones, but my main concern in this paper is located in the mathematical tradition of constructive mathematics from the Greeks to Fermat, Gauss and Kronecker, and from the logical side, in the finitist doctrine of Hilbert and his followers.

The philosophical outcome of mathematical constructivism may look as a meagre result to the Hegelian philosopher, but she should be reminded that mathematics, and particularly number theory, has been deemed as the most certain science « episteme» by Aristotle and Hilbert himself has put the emphasis on « certification » or <*Sicherung* > of his finitist proof theory to guarantee the same certainty to the contentual logic-- which I prefer to call internal logic-- he wanted in his metamathematics.

Hilbert uses the expression < *das inhaltliche logische Schliessen* >[1] which I translate by "internal logic", rather than logic of content. Brouwer and H. Weyl[2] use also the expression to designate an inner logic different from formal (external) logic which mirrors only the superficial structure of mathematics. For Hilbert, internal logic is not ordinary or formal logic, the rôle of which is only ancillary, that is the demonstration of theorems in a given mathematical theory. But internal logic, often identified with metamathematics[3], should be considered as an "intramathematics" in the sense that the inner consistency of axioms is more important than the deduction of particular theorems. In other words, proof theory <*Beweistheorie*> or <*Metamathematik*> is an internal logic to the extent it describes the inner workings of a mathematical theory. Proof theory has been seen as the theory of formal systems and, by extension, as the very embodiment of formalism. The hypothesis that I want to defend goes the other way : internal logic is the opposite of formalism and Hilbert's endeavour or programme could be formulated in the following terms : internal finitary logic reduces infinitary formal logic in the same

---

1 Cf. D. Hilbert (1930).

2 Weyl uses the term "intrinsic" which is very close to our "internal":

> Each field of knowledge, when it crystallizes into a formal theory, seems to carry with it its intrinsic logic which is part of the formalized symbolic system and this logic will, generally speaking, differ in different fields (Weyl, 1968, III, p. 705).

<*Inhaltlich*> has been translated sometimes by "contentual", it could also be rendered by "concrete" or "substantive". Beyond stylistic reasons, my use of "internal" is pointed and refers to a foundational approach which I have attempted to justify elsewhere (see Gauthier, 1991).

3 Cf. D. Hilbert (1932, III p. 174). For Hilbert's program, see G. Kreisel (1958) who does not however mention Kronecker's influence.

manner that a finitary mathematical theory (like arithmetic) reduces the infinite problems of the theory of forms or the theory of invariants to a finite calculus.

That hypothesis relies heavily on the assumption that Hilbert has been inspired by Kronecker's mathematical practice, especially by his fundamental work *Grundzüge einer arithmetischen Theorie der algebraischen Grössen* (« Foundations of an arithmetical theory of algebraic quantities »). My contention is, that despite his rare admission of a Kroneckerian influence (see below), Hilbert saw Kronecker's work as a model of mathematical practice, not as a categorical imperative of philosophical import. Hardly a constant adherent to Kronecker's finitism, he nevertheless stressed the importance of finiteness results and the constructive content of mathematical results. My hypothesis, in this attempted reconstruction of Hilbert's programme, is that despite his opposition to Kronecker's anti-Cantorianism, he wanted to save ideal structures (in a dialectical retreat from Brouwer's exclusivist attitude) by granting them a kind of ideal existence, that is consistency.

Hilbert's most important results must be replaced in the mathematical tradition he has inherited, the tradition of Gauss and Kronecker and I want to put the emphasis mainly on Kronecker who has inspired much of Hilbert's mathematical work. It is worth noting at first that Hilbert puts at the very foundation of his entreprise, the theory of finite intuitive arithmetic (arithmetical sentences without quantifiers); then follow quantified arithmetic sentences (with $\exists$ or $\forall$) which introduce an infinite (denumerable) number of elements, *e.g.* Euclid's theorem of the infinity of primes, Fermat's last theorem, etc., all theorems which are not immediately subjected to negation since they refer to the entire sequence (the set) of natural numbers, and finally, the transfinite mathematical statements which are transarithmetical by definition and which one must consider as ideal structures, much alike Kummer's ideal numbers, or more apropriately, as we shall see, as Kronecker's indeterminates <*Unbestimmte*>. In order to save Aristotelian logic, that is ordinary formal logic, Hilbert introduces a formalised language preserving classical laws of quantification for infinite arithmetical statements and for transfinite or transarithmetic statements. What Hilbert had sooner seen as formal logic was only the usual logic of ordinary mathematics interpreted as formal (external) calculus. But one had to go further to account for the internal character of intuitive finite arithmetic; from there, it should be possible to conceive an extension to the internal logic of arithmetic, that is a transarithmetical logic which could encompass the whole of mathematics. But the extension had to be conservative, *i.e.* the laws of arithmetic must remain valid and for that reason a consistency proof of infinite arithmetic (and analysis) was necessary.

Since finite intuitive arithmetic is self-consistent — here Hilbert concurs with Kronecker as is evident from Hilbert's early independence results in geometry and later in his foundational work — and immediately justified in intuition <*Anschauung*>, extended consistency has a conceptual <*begriffliche*> character that can be secured only by means of logic. Once consistency is obtained, ideal existence is warranted. I contrast here effective existence (of constructions) with ideal existence (of structures); the passage between the two is achieved by logic alone (what Hilbert called Aristotelian logic). Of course, the logic is non-constructive, but it must have a finitary embodiement, and that will be the task of finitist metamathematics conceived as an instrument for a consistency proof of analysis and set theory. The concepts of justification or certification, <*Sicherung*>, surveyability <*Uebersehbarkeit*>, are suposed to garantee the finiteness enjoyed by intuitive arithmetic. If this analysis is right, it shows that Hilbert's strategy for the consistency problem had to be motivated by a foundational approach akin to Kronecker's theory of arithmetic.

2. ARITHMETIC

Hilbert admires Kronecker's work in arithmetic, but he disapproves of his contempt for Cantor, whom Kronecker condemned as « perverter of youth ». As Hilbert declared : « nobody will drive use from the paradise Cantor has created for us »[4], and despite what Kronecker has said about the integers as creations of God[5], there is no doubt that Cantor's paradise is more populated than Kronecker's. However, it is not divine inspiration that one finds in Kronecker, but Gaussian ideas, when he says that number is a creation of our mind, while space and time have an independent reality that cannot be determined *a priori* or in an absolute[6] fashion. Kronecker here follows Gauss and Riemann against Kant. But mathematics is the work of a finite mind and constructive methods - explicit solutions - must replace existence theorems as in the fundamental theorem of algebra where an algebraic equation without roots (solutions) leads to a contradiction. Hilbert will listen to Kronecker in his arithmetical works, but he will turn a deaf ear when he is able to travel the transcendental royal road of existence theorems in invariant theory.

Already in his works on number theory, Hilbert shows some reluctance to Kummer's and Kronecker's arithmetical spirit. In his report on *The theory of algebraic number fields*, Hilbert says :

> I have attempted to bypass Kummer's heavy aparatus of calculation in order to abide by Riemann's precept, that is to obtain results through concepts and not by calculation[7].

Modern mathematics stands under the sign of number « *unter dem Zeichen der Zahl* »[8] and the arithmetization of function theory (analysis) is meant to show that the proof of a mathematical fact is ultimately reducible to relations among rational integers[9]. Kronecker would not have said differently and the indeterminate coefficients or simply indeterminates <*Unbestimmte*> which he introduces in 1881 are algebraic quantifies (independent variables) playing the role of ideal extensions[10]. It is in logic that Hilbert will introduce ideal entities in order to preserve logical laws for the transcendental world of the transfinite, but only as a detour <*Umweg*> and only to eliminate them after having introduced them.

---

4 See Hilbert (1926, p. 170).

5 Cf. D. Hilbert (1932, I, p. 64 and 1932, III, p. 161). Hilbert adds that Kronecker had rejected everything that transcended the integers.

6 Cf. L. Kronecker (1968, II, p. 249-274).

7 Cf. L. Kronecker (1968, I, p. 67). Helmut Hasse adds that Hilbert has given new proofs free of Kummer's detailed and opaque calculations, (*idem* , p. 259).

8 Underlined by Hilbert (1932, I, p. 66).

9 (*Idem* , p. 66).

10 Cf. L. Kronecker « Grundzüge einer arithmetischen Theorie der algebraischen Grössen » (1968, II, p. 237-387).

3. LOGIC

When Hilbert, in his conference « Ueber das Unendliche »[11] (« On the infinite »), explains that from a finitary point of view <*finiter Standpunkt*> there are two kinds of formulas in mathematics, the first ones corresponding to finitary statements and the second ones to ideal structures — which are deprived of meaning <*sinnlos*> — he simply translates Kronecker's language of a pure arithmetic and its indeterminate extensions (with ideal elements) into the metamathematics or proof theory he hopes to build. But if the extra-mathematical operations of logic are meaningless, as are non-algebraic quantities beyond the domain of rationality and if arithmetic alone is internal, algebra being purely formal, the formal system of logical operations will have only the role of a meaningless extension of arithmetic; but that extension will have to be consistent, that is after having eliminated the ideal structures (or the indeterminates), the validity of logical laws of elementary arithmetic or of the pure arithmetic of domains of rationality must be preserved. Logic must insure the passage from finite arithmetic to transfinite arithmetic (and analysis) and Hilbert's logical choice function is designed to fill the gap between the two.

The concrete objects that are meant to replace integers in Hilbert's metamathematics are simply signs and the finite combinatorial system they generate is the formal counterpart of arithmetic. « At the beginning is the sign », such is Hilbert's philosophical motto in 1922[12]. On that finitary basis, existing mathematical theories can be formalized by welding together logic and arithmetic. Such an arithmetical logic, as one could call it, hides an internal logic which beyond the formal proofs of ordinary mathematics must give access to a consistency proof of mathematics, since the subject matter of metamathematics is the total system of proof structures of usual mathematics. "Internal" logic must produce new axioms while formal logic only derives new theorems from known axioms.

The Hilbertian definition of a formal system with connectives and quantifiers is well known. Universal and existential quantifiers are defined with the help of a transfinite choice function $\varepsilon(A)$ which associates on object to each predicate or a number to each function

$$A(a) \rightarrow A(\varepsilon_x A(x)).$$

The universal quantifier is defined

$$\forall x A x \equiv (\varepsilon_x \neg A(x))$$

and the existential quantifier

$$\exists x A x \equiv A(\varepsilon_x A(x)).$$

---

[11]Cf. D. Hilbert (1926).

[12] Cf. D. Hilbert (1932, III, p.163). The number sign <*Zahlzeichen*> became later a <*Ziffer*> or number figure.

Mis en forme : Gauche, Retrait : Gauche : 0 cm, Première ligne : 0 cm

Two principles must apply

$$\forall x A x \rightarrow A(a) \qquad \text{(Aristotelian axiom)}$$[13]

and

$$\neg\forall x A x \rightarrow \exists x \neg A(x) \qquad \text{(excluded middle).}$$

Although the choice function is non-constructive, Hilbert believed that by employing it a finite number of times (a finite search), the finiteness of the procedure could be assumed and a consistency proof made possible. We know now that it was not the case as Gödel's incompleteness results have shown. Of course, Peano arithmetic, to which apply Gödel's results, contains more than Kronecker arithmetic which one could relate to predicative arithmetic[14], where the upper or logarithmic bounds are a distant echo of Kronecker's field theory.

Kronecker's genetic point of view has made it possible for him to escape the infinitary formal viewpoint. Hilbert, on the other side, tried to rescue ideal structures within a finitary framework borrowed to a large extent from Kronecker's (finitist) standpoint. Predicative arithmetic requires upper or logarithmic bounds in the same way as they are required in the Hilbert's theory of complete systems of invariants akin to Kronecker's field theory of entire algebraic functions.

The predicative point of view, may it be formalist or nominalist, is closer to Kronecker's than to Hilbert's viewpoint. As a matter of fact, predicative arithmetic can assimilate non-standard integers (infinitesimals) $\nu = \infty$ much in the same manner of Kronecker's indeterminates and there is even a passage from internal to external in Nelson's internal set theory, but this time, the theory is not predicative[15]. Only a predicative logic for predicative arithmetic would have seemed adequate for Kronecker.

Hilbert's formalism is thus a simple infinitary (or indeterminate) extension of finite constructivism *à la* Kronecker. Intuitive or internal (in the extended sense) truth of arithmetic grants it with the status of a radical arithmetical logic at the very foundation(s) of the mathematical entreprise.

Despite his numerous attacks against Kronecker, whom he characterizes at times as « dictator of interdictions » <*Verbotsdiktator*>, Hilbert has admitted in 1930 that :

> Kronecker has clearly formulated a conception wich he has made explicit in numerous examples : his conception corresponds essentially to our finistist viewpoint[16].

---

[13] Here I follow P. Bernays (1968, III, p. 190-216).

[14] Cf. E. Nelson (1987).

[15] Cf. E. Nelson (1977).

[16] Cf. D. Hilbert:

> …. hat Kronecker eine Auffassung klar ausgesprochen und durch zahlreiche Beispiele erläutert, die heute im wesentlichen mit unserer finiten Einstellung zusammenfällt…

quoted by P. Bernays in (1932, III, p.203). But Hilbert adds (1930, p.487) that Kronecker's error has been the banishment of infinitary (transfinite) proof methods — transfinite means here literally "beyond the finite" and evokes only by extension Cantor's theory of transfinite sets. As a matter of historical accuracy, the quotation by Bernays is taken literally from Hilbert's lecture *Die*

Hilbert's finitism is finally very close, through Kronecker, to Brouwer's intuitionism and Poincaré's semi-intuitionism. Finitism is not affected or contradicted by infinitary incompleteness results, it is only the infinitary formalist extension with the idea of absolute consistency which is doomed. And it is no surprise that infinite induction (or Peano's induction postulate) is at stake here. Gentzen's proof for the consistency of arithmetic invokes transfinite induction up to ε_0, the limit of the ω hierarchy

$$lim\, \omega = \varepsilon_0.$$

Peano's induction postulate is not predicative, the less so its transfinite extension. The internal logic of arithmetic[17] requires bounded induction, an "effinite" sequence, *i.e.* potentially infinite sequence of natural numbers and no more. It is somewhat paradoxical that it is the incompleteness results (admittedly gained from a transarithmetical viewpoint) that give meaning to the infinitary extensions of arithmetic Hilbert had in mind. More generally, metalogical results (including completeness theorems) are, in some sense, the only road that could be taken beyond the limits of finitary constructivism (of which Gödel was well-aware). In view of the great richness of results in non-standard arithmetic and analysis in model theory, on the one side, and feasibility arguments, definability problems below and inside the hyperarithmetical hierarchy as well as the various complexity (algorithmic and other) questions, on the other, metalogic seems to be, after Gödel's breach of the peace, the continuation of Hilbert's programme by other means; witness Gentzen's and Gödel's proofs of the consistency of arithmetic and the work of Kreisel and others in reductive proof theory, as well as independence and relative consistency results in set theory.

The ideal of consistency is nonetheless still here : to have access in analysis (and set theory) to the certainty <*Sicherheit*> possessed by finite arithmetic, the ultimate intuitive foundation. It was the same certainty that was supposed to guide metamathematics in its extended logic[18]. The question whether relative consistency results and non-standard models from a realist perspective or reductive proof theory and predicative theories (along with the various constructivisms) from a constructivist viewpoint are reconcilable, for the first with Hilbert's programme and with Kronecker's arithmetical ideal for the second, remains a philosophical question of importance. Meanwhile, Hilbert with his programme has fathered, by wind and tide, from Herbrand to Gödel and from Tarski to Robinson, the whole of modern logic.

4. FINITISM

Hilbert's programme can be modified, as Kreisel has suggested, it can be extended as Gödel supposed and it could be relativised in various directions as Feferman and Nelson

---

Grunglegung der elementaren Zahlenlehre , *Math. Annal.*, 104, 4, (1930). It is a sort of testament of uncertain legacy, since it takes to task Kronecker, Dedekind, Poincaré, Russell *et alii* on the excluded third and other topics to conclude that the subject "foundations of mathematics" has been eliminated from mathematics. There is no doubt that Hilbert meant "logical foundations", for proof theory is essentially an arithmetization of logic, which favors an internal arithmetical logic alien to any kind of "external" foundations, may they be logical or philosophical.

17 I refer here to my book (see Gauthier, 2002).

18 In geometry, consistency means simply to find an arithmetical model, arithmetic being self-consistent; that is the lesson of Hilbert's independence proofs in geometry.

have proposed. I have attempted rather to radicalize Hilbert's programme by founding it on Kronecker's programme. When Hilbert in his talk « Ueber das Unendliche » (1926) explains that, from the finitist stance <*finiter Standpunkt*>, there are two kinds of formulas in mathematicss, the ones that correspond to finitary statements and the ones that refer to ideal (meaningless) structures, he is just translating Kronecker's language of a general arithmetic whith its (indeterminate) extensions — which cover ideal elements — into metamathematics or the theory of proofs he wants to formulate. But the extra-arithmetical operations of logic are as meaningless as the algebraic quantities outside domains of rationality and if arithmetic alone is internal — algebra is formal or external — the formal system of logical operations will have the limited function of an extension of arithmetic, provided that such an extension is consistent, that is, once the ideal structures (or indeterminates) are eliminated, the validity of the logical laws of the primitive domain of arithmetic, in Kronecker's terms, the arithmetic of the natural domain of rationality is preserved. One sees immediately the close parallelism between Kronecker's programme and Hilbert's programme. The relationship is so striking that one could suppose that Hilbert is constantly inspired, consciously or not, by Kronecker's arithmetical constructivism.

The concrete objects that are going to replace integers in Hilbertian metamathematics are the signs and symbols of a finite combinatorics which is the formal counterpart of internal <*inhaltliche*> arithmetic or arithmetic with arithmetical content. At the beginning is the sign, this is Hilbert's philosophical motto as early as 1902. On a finitary basis, says Hilbert, existing mathematical theories can be formalized by the joint construction of logic and arithmetic. The resulting "arithmetical logic", as I call it, contains an internal (metamathematical) logic, which goes beyond the formal proof of ordinary mathematics, and leads to a proof of the non-contradiction of mathematics, since the object of metamathematics is the totality of proofs in usual mathematics. It is clear from Hilbert's pronouncements that there is a direction forward from arithmetic to set theory and analysis and while the ground arithmetical logic produces new axioms, formal logic only proceeds to the derivation of new theorems from existing axioms. Finitary logic suffices to warrant the intuitive validity of elementary arithmetic, but traditional logic should be able, Hilbert assumes, to extend that validity beyond elementary arithmetic. Hilbert then defines connectives and quantifiers accordingly using a transfinite choice function $\varepsilon(A)$ which associates an object to each predicate and a number to each function; thus, the universal quantifier is defined by a choice function which cannot find a counterexample to a given predicate (or to the image of a given function). Hilbert adds the Aristotelian axiom for the existential import of the universal quantifier and the principle of excluded third which means that negation of the universal quantifier implies the existence of a counterexample.

Although the choice function is not constructive, Hilbert believed that its iteration or reiteration a finite number of times secured the finite character of the procedure and that a consistency proof along those lines was certainly possible. Ackermann, as is well known, has succeeded in giving a consistency proof of arithmetic with the Hilbertian ε-substitution method (and with transfinite induction).

Hilbert's programme has failed because of Gödel results, but more importantly it has failed because it has deviated too far away from Kronecker's original programme. Kronecker had resisted the infinitist temptation by keeping close to arithmetic and if

Hilbert has yielded to the temptation, it is due to his submission to the (presumed) existence of ideal elements or to the formal definiteness of indeterminates, as one could say, the final elimination of which he could not achieve in his attachment to Cantor’s paradise. Hilbert’s formalism or rather the formal extension of finitist mathematics is but the non-finitist enlargment of the finitist position <*finite Einstellung*> and the dissolution of absolute consistency in relative consistency. It is not surprising, in retrospect, that it is the infinite induction of set-theoretic (Peano) arithmetic which is the heart of the matter. Hilbert in 1930 is still reproaching Kronecker with is rejection of infinitary proof methods and it is an ironic dramatic surprise that Gödel published, a year after Hilbert’s paper (1930) his incompleteness proof for Peano arithmetic using a method of proof which can be said infinitary, since it uses Cantor’s diagonal procedure on the infinite set of natural numbers.

5. CONSTRUCTIVISM

Internal logic is the logic internal to mathematical discourse, primarily the arithmetical discourse. The logic in question tends to vanish as a component of arithmetic and is readily identified to the inferential structure of arithmetic. Internal logic becomes arithmetical or polynomial logic — or modular logic as we shall say later on. The internal structure can be exhibited with the help of ordinary logic (Hilbert says Aristotelian logic) or intuitionistic (constructive) logic. The internalization of logic in the case of arithmetic means the arithmetization of logic, that is the polynomial interpretation of logic which I have achieved on the model of Kronecker’s general arithmetic. I have claimed that Hilbert’s programme was conceived originally along the same line of thought. The idea that consistency (and decidability) were internal properties of mathematical theories was Hilbert’s own motive in his first attempts at defining the consistency problem — which had to be solved, as we have seen, in terms of polynomial equations.

Internal is intimately related to constructivist foundations and it is important to look into the body of evidence for construction.

First of all, constructivist foundations can be contrasted with structuralist foundations. The Bourbaki school, the many-headed hydra of mathematics, has inspired some recent work in philosophy of mathematics and philosophy of science. Resnick, Shapiro and others on the one side and Stegmüller, Sneed and others (in particular, Günther Ludwig) on the other have put much emphasis on the purely structural dimensions of mathematics or physics. Without being overcritical, one should be more careful in adopting the structuralist approach, not only because Bourbaki held a strict formalism viewpoint, but because the concept of "structures" had to do more with an axiomatic reconstruction than with actual practise. C. Ehresmann was probably the first mathematician to use extensively the term "local structure" : it means essentially topological space. Structuralism is in more ways than one a question of language and it is certainly a mistake to see in structures some kind of ideal objects in a (originally) French-speaking Platonist paradise. As far as science (mainly physics) is concerned, the structuralist approach is limited to the anatomic description of the skeleton of a theory in a set-theoretic setting. The semantics of the poorest of physical theories can only enrich it with borrowed ontological furniture, that is the semantical universe of sets. If the idiom of structures is at all useful, it is in the province of philosophy, where one is far enough from theory and practise to reconstruct a simplified picture of the relationships between theory and practise. Foundational studies aim at bridging the gap or at building the bridge, as Hilbert would say, between the two and constructivist foundations come closer to the footings. Constructivism is not prrimarily interested in the (historical or psychological) genesis of concepts; it is concerned with the process of construction of mathematical concepts, (including physical and other scientific concepts) and their constructive content. I propose to make the distinction between structural concepts and constructive concepts : structural concepts would be global, first-approximation concepts reminiscent somehow of Kant's <*Grenzbegriffe*> or limit-concepts, while constructive concepts would be local, explicit, "procedural" concepts which do not describe an ideal "state of affairs", but the actual process for which the concept is only a stenogram. Take the concept of set, it should be considered as a stenogram for the process of setting or putting together some (concrete) things or analogues thereof; it would be clear then that the concept of set can only apply to finite collections or "set-tings". In that line of tought, structures are only products of constructions. The distinction local-global is pervasive in mathematics, from differential geometry to algebraic geometry (topoi theory), from group theory to algebraic number theory. In his *Zahlentheorie*, H. Hasse emphasizes the distinction between an algebraic number field K or an extension thereof K/m and localisation or singularisation of K, K_p where p is a particular prime number; the study of the former is global, <*im Großen*>, "in the large", while the study of the latter is local, <*im Kleinen*>, or in the small. Our distinction, admittedly more general, shares the same spirit of distinctiveness.

6. CONCLUSION

Is a "local logic" possible at all? If logic is to be internalized, it must be to some extent context-dependent; in the case of arithmetical logic, it means that the logic is minimal and corresponds to the internal structure of arithmetic and that logical operations are to be identified with arithmetical operations in a general arithmetic. Traditionally, logic has been viewed as the most general "facility" (with grammar) "serving all comers" in the words of Quine (1970).

For that purpose, the simplest of foundations would do the job; for logic, it would be first-order logic, for mathematics, (first-order) set-theory — or category theory —, for physics, some first-order language capable of expressing all the true assertions of physical theory. Where does the privilege of first-order language come from? In a world of properties and qualities, higher-order languages seem natural. My contention is that properties, qualities and the corresponding entities are all structural concepts in want of construction. The first-order concepts will be (constructive) approximations of second or higher-order structural concepts. The fact that they are only approximations does not mean that first-order concepts are not good enough, it means that the ideal structural concepts are vague, indeterminate and that the exact, effective, constructive concepts cannot capture the vagueness and globality of the higher-order concepts. Examples of those concepts are the concepts of arbitrary subset of an infinite set, the concept of fundamental sequence (as arbitrary or unnatural as the first one) or arbitrary predicate in (categorical) second-order number theory; such concepts are irretrievably structural. Take, on the other side, the concepts of a real-valued function and of an analytic function : they are representable respectively by trigonometric series and by power series. Recall also that generally in constructive mathematics, one considers cardinality $\aleph_0$ as the expression of the first-order, $2^{\aleph_0}$ corresponding to $P(\omega)$ as the second order of the universe of transfinite sets — Bishop, Markov, Shanin and others (sometimes even Dieudonné) have put to use that general idea. Another theme would be the traditional idea that geometry could be arithmetized, geometry itself being originally constructive. Of course, constructivist foundations are not limited to the interplay of geometry and arithmetic or to logic and mathematics : there should be a constructivist physics, a constructivist science (including biology, psychology, etc.). But such an ambitious programme cannot be carried out in a single stroke (or by a single man). I shall restrict myself to the outlines of the foundational scheme.

The basic question of internal logic is obviously the internality : it is not be conceived as a remnant of essentialism, in which the essence, the inner core of discourses (or words) or things would be revealed as possessing an intrinsic logic, as if an universal logos pervaded the integrity of the cosmos. Rather, internal can be taken as synonymous with local, that is particular, special or restricted. Logic is internal in many ways and universality is nothing more in that perspective than an integration process of different logics. That there is a basic logic is a foundational hypothesis that one can only test and verify (or falsify). The measure of success or failure is experimentation, in this as in other scientific enterprises.

"Arithmetical logic" is an appropriate name for the basic logic I imagine. Not a logic based on arithmetic, but the logic of arithmetic, essentially, operations of some agent and processes objectivized in some minimal language. The operations are the arithmetical ones and the processes are what I have come to call effinite sequences (finite sequences are simply sets). Minimality indicates that the arithmetical logic has a maximum of constraints, other logics will relay those constraints up to, maybe, a maximally simple Boolean logic. Within the framework of arithmetical logic, one has the usual operations of addition, subtraction, division, multiplication, exponentiation and infinite descent for the inductive procedure. The logical constants are interpreted as equivalent to arithmetical operations, disjunction as addition, conjunction as multiplication, implication as exponentiation, negation or local complementation as substraction, the existential quantifier as a numerical statement; the universal quantifier applies to finite integral domains (sets) and the effinite quantifier Ξx applies to effinite sequences. Domains of arithmetical statements are local (or locally finite) as are their exteriors (of negated statements interpreted in the negative integers) or congruences in a modular logic.

The main point is that arithmetical logic serves as a foundation for all logics, beginning with propositional logic and first-order predicate logic. This is a reversal of perspective, a non-Fregean, anti-logicist turn. Traditionally, arithmetic has been used as an interpretative structure; the arithmetic universe and the set-theoretic universe were companions in formal semantics. Take Gödel’s incompleteness results : arithmetization of syntax proceeds from the representability of arithmetic in a given formal system of arithmetic (Peano’s) and Tarski’s model theory needs the arithmetic universe for the notion of satisfaction. Hilbert for his proof of the independence of Euclid’s fifth axiom, Herbrand for his notion of the "reduced field" and Skolem in his finitary foundations of arithmetic did not think otherwise. Before the advent of logic, arithmetic played a similar rôle in most mathematical disciplines, analysis, the arithmetization of analysis (Weierstrass, Dedekind) and algebra (Kummer, Kronecker). The contemporary history of mathematics is no less illustrative of my point of view but goes further and motivates my present approach (see Gauthier, 1978).

The arithmetical proofs of Selberg, Erdös and others of the prime number theorem and Dirichlet's theorem on arithmetic progressions are a landmark in arithmetic or arithmetic number theory — to distinguish it from analytic number theory and algebraic number theory. Not only because elementary, that is arithmetical, means as opposed to transcendantal or analytical method, are employed in those proofs, but mainly because the constructive content of the arithmetical statements is extracted from the structural gangue. I visualize the degrees of constructivity in the following order finite $\rightarrow$ elementary $\rightarrow$ arithmetical $\rightarrow$ effective $\rightarrow$ constructive — one could probably add combinatorial and computational or algorithmic and put an equivalence sign between some members of that list. Once again, the techniques of approximation here mean only that the structural concepts (and existence theorems) involved in transcendental methods are not totally accessible to constructive methods. But the process of emptying the mathematical paradise is an ongoing task.

Contemporary algebraic geometry, exhibits also some of the features of what I have called (not ironically) arithmetic number theory. Algebraic geometry is mostly complex algebraic geometry : real algebraic geometry has not the extent and the interest of the former. In view of recent results by Deligne on Weil's conjecture (for rational varieties) and Faltings on Mordell's conjecture (the finite number of rational points on an algebraic curve), there exists today what is called arithmetic-algebraic geometry (cf. G. Faltings et al. *Rational Points*, 1984). One could argue that those results have been obtained by using a variety of methods, including analytical-transcendental ones. But my contention is that arithmetization here has the same motivation and the same aim as the arithmetization of analysis, that is find effective bounds (classically, the notion of limit) for finiteness results (classically, continuity conditions) by construction in the proper sense of the word. There are equivalent formulations for computable, quantitative or explicit bounds. Notice that there abound proofs by contradiction, but they are logically "harmless" from a constructive viewpoint, since they derive most of the time from Fermat's method of "infinite descent", which is rather finite, terminating at 1 unlike Gentzen's transfinite induction which goes up to ε_0. I have termed "transduction" the process of going over ω, because it transcends the (effinite sequence of) integers; infinite descent could also be called more appropriately "reduction", rather than descending induction, not unlike the reduction procedure that is used in arithmetic algebraic geometry. Beyond the mere question of terminology, it is the central problem of "effectivity" which is at stake here and it is more than ever at the very heart of contemporary mathematics; logical analysis is called upon when "transcendental" surgery is needed.

But logical analysis must be careful : the choice of concepts and the employement of methods are a subtle enterprise, as Kreisel has constantly emphasized. And if the distinction between elementary and analytic proofs pertains more to philosophy than mathematics (cf. W. J. Ellison *Les nombres premiers*, 1975), it is not the case that effective methods and the inherent constructive attitude are pushed in the backyard of mathematical research, to wit rational approximations to algebraic numbers in number theory and the various finiteness results in algebraic geometry. One general feature that ought to be analysed in that context is what one could call "free induction", a procedure that uses "freely" the axiom of choice or some analogon as a creative procedure in mathematics. By the axiom of choice or its equivalents one obtains indeed bounds, but they are not effective — Hilbert's elimination of the ε-symbol and recursive function theory did the job for some time. Analytical continuation, most existence theorems and many other concepts seem to rest on the analogy between restricted induction (over natural numbers) and free induction or "transduction" over other (non-integral) domains. One could probably show that such use of "unnatural" induction is the source of most of the ineffective or transcendental methods in mathematics. I should be noted that algebraization or algebraic verifications, as Weyl would say (as in group theory *e.g.* Weyl's results on semi-simple groups), has the same (computational and procedural) constructive meaning.

Analytical-arithmetical statements, like the prime number theorem, seem to require trans-arithmetical methods — Hadamard proved the prime number theorem in 1896 by using the transcendental theory of entire (complex) functions. But constructivization has shown to be possible and it is this vein that I want exploit for foundational reasons.

But logic is the usual sense would then be derived from arithmetic as a (processual) combinatorics of ordinary language — the theory of language would have to show how meaning has a combinatorial nature, an idea which is not foreign to Hjelmslev's linguistic theory and Chomsky's idea of a recursive grammar. But more importantly, the incompleteness of arithmetic would be reflected in first-order logic and no artificial completeness proof (in an infinite integral domain) would be possible. Again completeness would become an experimental matter (for fragments of arithmetical logic), not a natural consequence of the model-theoretic approach. Naturalness in that context points to Lindström's theorem — first-order logic is the only logic closed under completeness, compacity and Löwenheim-Skolem theorems — and perhaps to the framework of classical logic and admissible set theory. The inversion of arithmetical logic discards those abstract benefits. Naturalness points also to second-order models, categoricity and non-standards models. "Natural" means the principal model of arithmetic with the "natural" second-order induction postulate : but it is once more the model-theoretic idiosyncrasy which is postulated as the "natural" context associated with the structural vagueness or globality of intensional concepts.

A recent result in arithmetic, Paris' and Harrington's theorem on an undecidable arithmetical statement of a combinatorial nature within Peano's arithmetic, not a logical undecidable (diagonalized) statement, is another indication of the primacy of arithmetic. Arithmetical logic does not privilege arithmetic over logic, it turns logic upside down, deprives it of its semantical metalogic and gives it the status of an "operational" or "procedural" theory of inference and validity. The combining of statements (their formation and transformation according to logical form) becomes a regulative process entirely determined by the arithmetic art, thus realizing (hopefully) Leibniz's dream for a programme less ambitious than an <*ars combinatoria*> for the whole universe of concepts.

Yvon Gauthier
University of Montreal

REFERENCES

Ellison, W. J.,
(1975) *Les nombres premiers*, Hermann, Paris.

Gauthier, Y.,
(1991) *De la logique interne*, Vrin, Paris.
(2002*) Internal Logic. Foundations of Mathematics from Kronecker to Hilbert*, Kluwer, “Synthese Library”, Dordrecht/London/Boston.

Hilbert, D.,
(1904) « Ueber die Grundlagen der Logik und der Arithmetik », in Hilbert (1935).
(1926) « Ueber das Unendliche », *Mathematische Annalen* 95, 161-190.
(1930) « Die Grundlegung der elementaren Zahlenlehre », *Mathematische Annalen* 104, 485-494.
(1932) *Gesammelte Abhandlungen*, 3 Bände, Chelsea, New-York.
(1935) « Neubegründung der Mathematik » and « Die logischen Grundlagen der Mathematik », in *Gesammelte Abhandlungen*, Springer-Verlag, Berlin, Vol. 3, 157-177 and 178-191

Hilbert, D. and Bernays, P.,
(1968) *Grundlagen der Mathematik I*, 2 Aufl., Springer-Verlag, Berlin.
(1970) *Grundlagen der Mathematik II*, 2 Aufl., Springer-Verlag, Berlin.

Kreisel, G.,
(1958) « Hilbert’s Programme », *Dialectica* 12, 346-372. Revised with a "Postcript" in (1983) *Philosophy of Mathematics*, by Benacerraf and H. Putnam (eds.), 2nd ed., Englewood Cliffs, New Jersey

Kronecker, L.,
(1901) *Vorlesungen über Zahlentheorie*, vol. I, K. Hensel (ed.), Teubner, Leipzig.
(1968a) *Werke*, K. Hensel (ed.), 5 vols., Chelsea, New York.
(1968b) « Ueber den Zahlbegriff », *Werke* , vol II, 252-274.

Nelson, E.,
(1977) « Internal Set Theory : a New Approach to Nonstandard Analysis », *Bulletin of the American Mathematical Society* 83, 1165-1198.
(1986) *Predicative Arithmetic*, Mathematical Notes 32, Princeton University Press, Princeton, New Jersey.

Weyl, H.
(1968) *Gesammelte Abhandlungen*, 4. Bände, K. Chandrasekharan (ed.), Springer-Verlag, Berlin.

.